

Mean-square Convergence of a Symplectic Local Discontinuous Galerkin Method Applied to Stochastic Linear Schrödinger Equation[†]CHUCHU CHEN[‡], JIALIN HONG[§]*State Key Laboratory of Scientific and Engineering Computing, Institute of Computational Mathematics and Scientific/Engineering Computing, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, China*

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In this paper, we investigate the mean-square convergence of a novel symplectic local discontinuous Galerkin method in \mathbb{L}^2 -norm for stochastic linear Schrödinger equation with multiplicative noise. It is shown that the mean-square error is bounded not only by the temporal and spatial step-sizes, but also by their ratio. The mean-square convergence rate with respect to the computational cost is derived under appropriate assumptions for initial data and noise. Meanwhile, we show that the method preserves the discrete charge conservation law which implies an \mathbb{L}^2 -stability.

Keywords: symplectic method; local discontinuous Galerkin method; stochastic linear Schrödinger equation; \mathbb{L}^2 -stability; charge conservation law; mean-square convergence.

1. Introduction

In this paper we consider the stochastic linear Schrödinger equation with multiplicative noise

$$idu - (\Delta u + Q(x)u)dt = u \circ dW, \quad u(x, 0) = u_0(x), \quad (1.1)$$

where $t \in [0, T]$, $x \in \mathcal{O} \subset \mathbb{R}^d$, $Q \in \mathbb{H}^3(\mathcal{O})$ and periodic boundary condition holds. Here, the \circ in the last term in (1.1) means that the product is of Stratonovich type and W on $\mathbb{L}^2(\mathcal{O}^d)$ is a real-valued Wiener process with a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \in [0, T]})$. It has the expansion form $W(t, x, \omega) = \sum_{k=0}^{\infty} \beta_k(t, \omega) \phi e_k(x)$, with $(e_k)_{k \in \mathbb{N}^d}$ being an orthonormal basis of $\mathbb{L}^2(\mathcal{O}^d)$, $\{\beta_k\}_{k \in \mathbb{N}^d}$ being a sequence of independent Brownian motions and $\phi \in \mathcal{L}_2(\mathbb{L}^2(\mathcal{O}^d); \mathbb{H}^{\gamma}(\mathcal{O}^d))$ being a Hilbert-Schmidt operator. The phase flow of equation (1.1) is stochastic symplectic (see Chen & Hong, 2014), i.e.,

$$\bar{\omega}(t) = \int_{\mathcal{O}} d(r(t)) \wedge d(s(t)) dx = \bar{\omega}(0),$$

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with r and s being the real and imaginary parts of u , respectively, and its solution preserves the charge conservation law almost surely (see Bouard & Debussche, 2003), i.e.,

$$\int_{\mathcal{O}} |u(x, t)|^2 dx = \int_{\mathcal{O}} |u_0(x)|^2 dx.$$

We propose a symplectic local discontinuous Galerkin method to discrete equation (1.1) in order to on one hand preserve the properties of the original problems as much as possible and on another hand combine the attractive properties of local discontinuous Galerkin method (see, e.g., Cockburn & Shu, 1998; Cockburn *et al.*, 2000; Cockburn & Shu, 2001). We refer interested readers to (Xu & Shu, 2005) and references therein for the numerical simulation of the deterministic Schrödinger equation based on local discontinuous Galerkin method. Due to the reason that equation (1.1) is meaningful in the sense of integral, we apply the midpoint scheme to discretize the temporal direction at first avoiding dealing with double temporal-spatial integrals which is introduced by stochastic integral and local discontinuous Galerkin discretization. It is shown that the midpoint semi-discretization not only is a symplectic method, but also possesses the discrete charge conservation law. Furthermore, we show that the semi-discretization is of order 1 in mean-square convergence sense via a direct approach while authors in (Chen & Hong, 2014) proved the same result via a fundamental convergence theorem on the mean-square convergence for the temporal semi-discretizations. The main difficulty lies in the analysis of the mean-square convergence order for the spatial direction, where we use local discontinuous Galerkin method to discrete the semi-discretized equation and obtain the full-discrete method which is called symplectic local discontinuous Galerkin method in this paper. We solve it by means of the standard approximation theory of projection operator, Itô isometry and the adapted properties of processes u and W . As a result we analysis the mean-square convergence error for the symplectic local discontinuous Galerkin method, and derive the mean-square convergence rate with respect to the computational cost under appropriate hypothesis on initial data and noise. Moreover theoretical analysis shows that the obtained full-discrete method is \mathbb{L}^2 -stable and preserves the discrete charge conservation law.

The rest of this paper is organized as follows. In section 2, we propose the symplectic local discontinuous Galerkin method for stochastic Schrödinger equation and derive the discrete charge conservation law. In section 3, we study the mean-square convergence of the obtained method and present the mean-square error estimation. Some proofs and calculations are postponed to the final appendices.

2. The symplectic local discontinuous Galerkin method

In this section, we will apply implicit midpoint scheme to (1.1) in the temporal direction, then we discretize the spatial direction by local discontinuous Galerkin method and obtain the full-discrete method.

2.1 Temporal semi-discrete scheme

The midpoint scheme for (1.1) reads

$$iu^{n+1} = iu^n - \Delta t \left(\Delta u^{n+\frac{1}{2}} + Q(x)u^{n+\frac{1}{2}} \right) + u^{n+\frac{1}{2}} \Delta \tilde{W}_n, \quad n = 0, 1, \dots, N \quad (2.1)$$

where Δt is the time step size, $N = \frac{T}{\Delta t}$, $u^{n+\frac{1}{2}} = \frac{1}{2}(u^{n+1} + u^n)$, and $\Delta \tilde{W}_n = \sum_{k=0}^{\infty} \sqrt{\Delta t} \zeta_{k,n}^{\kappa} \phi e_k(x)$ with $\zeta_{k,n}^{\kappa}$ being the truncation of a $\mathcal{N}(0, 1)$ -distribution random variable $\xi_{k,n}$:

$$\zeta_{k,n}^{\kappa} = \begin{cases} \kappa & \text{if } \xi_{k,n} > \kappa; \\ \xi_{k,n} & \text{if } |\xi_{k,n}| \leq \kappa; \\ -\kappa & \text{if } \xi_{k,n} < -\kappa \end{cases}$$

with $\kappa := \sqrt{4|\ln(\Delta t)|}$. This choice is motivated by the fact that standard Gaussian random variables are unbounded for arbitrary values of Δt , (see Milstein *et al.*, 2002) for more details. For the truncated Wiener process, we have the following properties:

$$\begin{aligned} \text{(i)} \quad & E \|\Delta \tilde{W}_n - \Delta W_n\|_{\mathbb{H}^1}^2 \leq K \Delta t^3, \\ \text{(ii)} \quad & E \|(\Delta \tilde{W}_n)^2 - (\Delta W_n)^2\|_{\mathbb{H}^1}^2 \leq K \Delta t^4, \end{aligned} \tag{2.2}$$

where the constant K depends on $\|\phi\|_{\mathcal{L}_2(\mathbb{L}^2, \mathbb{H}^1)}$. Based on the fact that \tilde{W} is real-valued, by multiplying both sides of equation (2.1) by $u^{n+\frac{1}{2}}$, which is the conjugate of $u^{n+\frac{1}{2}}$, and then taking the imaginary part and integrating it over the whole space domain. We can get the discrete charge conservation law as follows.

PROPOSITION 2.1 Under the periodic boundary conditions, the semi-discrete scheme (2.4) of the system (1.1) has the discrete charge conservation law, i.e.,

$$\int_{\mathcal{O}} |u^{n+1}(x)|^2 dx = \int_{\mathcal{O}} |u^n(x)|^2 dx, \quad n = 0, 1, \dots, N. \tag{2.3}$$

Furthermore, the semi-discrete scheme (2.4) preserves the stochastic symplectic structure; (see Chen & Hong, 2014).

PROPOSITION 2.2 The implicit midpoint scheme (2.4) for the system (1.1) is stochastic symplectic.

2.2 Temporal-spatial full-discrete method

In this subsection, we consider the local discontinuous Galerkin method for the system (2.4) in the spatial direction and obtain the full-discrete method. To this end, we introduce some spatial-grid notations for the case $d = 1$, $\mathcal{O} = [L_f, L_r]$ for simplicity, and the results hold for the general dimensional problems. We denote the mesh by $I_j = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]$, for $1 \leq j \leq J$, where $L_f = x_{\frac{1}{2}} < x_{\frac{3}{2}} < \dots < x_{N+\frac{1}{2}} = L_r$. Let $\Delta x_j = x_{j+\frac{1}{2}} - x_{j-\frac{1}{2}}$, $1 \leq j \leq J$ with $h = \max_{1 \leq j \leq J} \Delta x_j$ being the maximum mesh size. Assume the mesh is regular, namely there is a constant $c > 0$ independent of h such that $\Delta x_j \geq ch$, $1 \leq j \leq J$.

By decomposing the complex function u^n into its real and imaginary parts $u^n = r^n + is^n$ with r^n and s^n being real functions, we obtain the following first order semi-discrete system

$$\begin{aligned} r^{n+1} &= r^n + \left((p_x)^{n+\frac{1}{2}} + Q(x) s^{n+\frac{1}{2}} \right) \Delta t + s^{n+\frac{1}{2}} \Delta \tilde{W}_n, \\ p^{n+\frac{1}{2}} &= (s_x)^{n+\frac{1}{2}}, \\ s^{n+1} &= s^n - \left((q_x)^{n+\frac{1}{2}} + Q(x) r^{n+\frac{1}{2}} \right) \Delta t - r^{n+\frac{1}{2}} \Delta \tilde{W}_n, \\ q^{n+\frac{1}{2}} &= (r_x)^{n+\frac{1}{2}}. \end{aligned} \tag{2.4}$$

We consider the local discontinuous Galerkin method for the system (2.4) in the spatial direction and obtain the full-discrete method: find $r_h, p_h, s_h, q_h \in V_h^k$, which now denotes real piecewise polynomial of degree at most k , such that, for all test functions $v_h, \omega_h, \alpha_h, \beta_h \in V_h^k = \{v : v \in P^k(I_j); 1 \leq j \leq J\}$ with $P^k(I_j)$ being the set of polynomials of degree up to k defined on the cell I_j .

$$\begin{aligned}
& \int_{I_j} r_h^{n+1} v_h dx - \int_{I_j} r_h^n v_h dx - \Delta t \left[(\hat{p}^{n+\frac{1}{2}} v_h^-)_{j+\frac{1}{2}} - (\hat{p}^{n+\frac{1}{2}} v_h^+)_{j-\frac{1}{2}} \right] \\
& + \Delta t \int_{I_j} \left(p_h^{n+\frac{1}{2}} (v_h)_x - s_h^{n+\frac{1}{2}} Q_h v_h \right) dx - \int_{I_j} s_h^{n+\frac{1}{2}} v_h \Delta \tilde{W}_n dx = 0, \\
& \int_{I_j} p_h^{n+\frac{1}{2}} \omega_h dx + \int_{I_j} s_h^{n+\frac{1}{2}} (\omega_h)_x dx - \left[(\hat{s}^{n+\frac{1}{2}} \omega_h^-)_{j+\frac{1}{2}} - (\hat{s}^{n+\frac{1}{2}} \omega_h^+)_{j-\frac{1}{2}} \right] = 0, \\
& \int_{I_j} s_h^{n+1} \alpha_h dx - \int_{I_j} s_h^n \alpha_h dx + \Delta t \left[(\hat{q}^{n+\frac{1}{2}} \alpha_h^-)_{j+\frac{1}{2}} - (\hat{q}^{n+\frac{1}{2}} \alpha_h^+)_{j-\frac{1}{2}} \right] \\
& - \Delta t \int_{I_j} \left(q_h^{n+\frac{1}{2}} (\alpha_h)_x - r_h^{n+\frac{1}{2}} Q_h \alpha_h \right) dx + \int_{I_j} r_h^{n+\frac{1}{2}} \alpha_h \Delta \tilde{W}_n dx = 0, \\
& \int_{I_j} q_h^{n+\frac{1}{2}} \beta_h dx + \int_{I_j} r_h^{n+\frac{1}{2}} (\beta_h)_x dx - \left[(\hat{r}^{n+\frac{1}{2}} \beta_h^-)_{j+\frac{1}{2}} - (\hat{r}^{n+\frac{1}{2}} \beta_h^+)_{j-\frac{1}{2}} \right] = 0.
\end{aligned} \tag{2.5}$$

In the sequel, we denote by $(u_h)_{j+\frac{1}{2}}^+$ and $(u_h)_{j+\frac{1}{2}}^-$ the values of u_h at $x_{j+\frac{1}{2}}$, from the right cell I_{j+1} , and from the left cell I_j , respectively. And the numerical fluxes become

$$\hat{p} = p^+, \hat{r} = r^-, \hat{q} = q^+, \hat{s} = s^-, \tag{2.6}$$

where we have omitted the half-integer indices $j + \frac{1}{2}$ as all quantities in (2.6) are computed at the same points.

REMARK 2.1 The choice for the fluxes (2.6) is not unique. The important point is that \hat{r} and \hat{q} , \hat{s} and \hat{p} should be chosen from different directions.

With such a choice of fluxes (2.6), we can get the first main result about discrete charge conservation law of the symplectic local discontinuous Galerkin method (2.5).

THEOREM 2.3 Under the periodic boundary conditions, the symplectic local discontinuous Galerkin method (2.5) has the discrete charge conservation law, i.e.,

$$\int_{L_f}^{L_r} |u_h^{n+1}|^2 dx = \int_{L_f}^{L_r} |u_h^n|^2 dx, \quad n = 0, 1, 2, \dots, N. \tag{2.7}$$

Proof. To complete the proof of the discrete charge conservation law. First, we write (2.5) as the complex form. Denote $u_h^n = r_h^n + i s_h^n$, $\psi_h^n = q_h^n + i p_h^n$, and take $\alpha_h = v_h$, $\beta_h = \omega_h$, then (2.5) become

$$\begin{aligned}
& i \int_{I_j} u_h^{n+1} v_h dx - i \int_{I_j} u_h^n v_h dx - [(\hat{\psi}^{n+\frac{1}{2}} v_h^-)_{j+\frac{1}{2}} - (\hat{\psi}^{n+\frac{1}{2}} v_h^+)_{j-\frac{1}{2}}] \Delta t \\
& + \Delta t \int_{I_j} \left(\psi_h^{n+\frac{1}{2}} (v_h)_x - u_h^{n+\frac{1}{2}} Q_h v_h \right) dx - \int_{I_j} u_h^{n+\frac{1}{2}} v_h \Delta \tilde{W}_n dx = 0, \\
& \int_{I_j} \psi_h^{n+\frac{1}{2}} \omega_h dx + \int_{I_j} u_h^{n+\frac{1}{2}} (\omega_h)_x dx - [(\hat{u}^{n+\frac{1}{2}} \omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{n+\frac{1}{2}} \omega_h^+)_{j-\frac{1}{2}}] = 0.
\end{aligned} \tag{2.8}$$

where

$$\hat{u} = r_h^- + is_h^-, \quad \hat{\psi} = q_h^+ + ip_h^+. \quad (2.9)$$

Now, we take the complex conjugate for every terms in system (2.8)

$$\begin{aligned} & -i \int_{I_j} (u_h^*)^{n+1} v_h^* dx + i \int_{I_j} (u_h^*)^n v_h^* dx - \Delta t \left[(\hat{\psi}^{*n+\frac{1}{2}} v_h^{*-})_{j+\frac{1}{2}} - (\hat{\psi}^{*n+\frac{1}{2}} v_h^{*+})_{j-\frac{1}{2}} \right] \\ & + \Delta t \int_{I_j} \left(\psi_h^{*n+\frac{1}{2}} (v_h^*)_x - u_h^{*n+\frac{1}{2}} Q_h v_h^* \right) dx - \int_{I_j} u_h^{*n+\frac{1}{2}} v_h^* \Delta \tilde{W}_n dx = 0, \\ & \int_{I_j} \psi_h^{*n+\frac{1}{2}} \omega_h^* dx + \int_{I_j} u_h^{*n+\frac{1}{2}} (\omega_h)_x dx - \left[(\hat{u}^{*n+\frac{1}{2}} \omega_h^{*-})_{j+\frac{1}{2}} - (\hat{u}^{*n+\frac{1}{2}} \omega_h^{*+})_{j-\frac{1}{2}} \right] = 0. \end{aligned} \quad (2.10)$$

We introduce a short-hand notation

$$\begin{aligned} \mathfrak{H}_j(u_h^n, \psi_h^n; v_h, \omega_h) &= i \int_{I_j} u_h^{n+1} v_h dx - i \int_{I_j} u_h^n v_h dx - \Delta t \int_{I_j} \psi_h^{n+\frac{1}{2}} \omega_h dx \\ &+ \Delta t \int_{I_j} \left(\psi_h^{n+\frac{1}{2}} (v_h)_x - u_h^{n+\frac{1}{2}} Q_h v_h \right) dx - \int_{I_j} u_h^{n+\frac{1}{2}} v_h \Delta \tilde{W}_n dx \\ &- \Delta t \int_{I_j} u_h^{n+\frac{1}{2}} (\omega_h)_x dx - \Delta t \left[(\hat{\psi}^{n+\frac{1}{2}} v_h^-)_{j+\frac{1}{2}} - (\hat{\psi}^{n+\frac{1}{2}} v_h^+)_{j-\frac{1}{2}} \right] \\ &+ \Delta t \left[(\hat{u}^{n+\frac{1}{2}} \omega_h^-)_{j+\frac{1}{2}} - (\hat{u}^{n+\frac{1}{2}} \omega_h^+)_{j-\frac{1}{2}} \right]. \end{aligned} \quad (2.11)$$

Then from (2.10), we also have the expression of $\mathfrak{H}_j^*(u_h^n, \psi_h^n; v_h, \omega_h)$. If we take $v_h = u_h^{*n+\frac{1}{2}}$, $\omega_h = \psi_h^{*n+\frac{1}{2}}$ in both functions $\mathfrak{H}_j(u_h^n, \psi_h^n; v_h, \omega_h)$ and $\mathfrak{H}_j^*(u_h^n, \psi_h^n; v_h, \omega_h)$, both functions are zero. Hence we obtain

$$\mathfrak{H}_j(u_h^n, \psi_h^n; u_h^{*n+\frac{1}{2}}, \psi_h^{*n+\frac{1}{2}}) - \mathfrak{H}_j^*(u_h^n, \psi_h^n; u_h^{*n+\frac{1}{2}}, \psi_h^{*n+\frac{1}{2}}) = 0. \quad (2.12)$$

With (2.9) of the numerical fluxes, then (2.12) becomes

$$\begin{aligned} & i \int_{I_j} \left(|u_h^{n+1}|^2 - |u_h^n|^2 \right) dx + \underbrace{\Delta t \int_{I_j} \left(\psi_h^{n+\frac{1}{2}} (u_h^{*n+\frac{1}{2}})_x + u_h^{*n+\frac{1}{2}} (\psi_h^{n+\frac{1}{2}})_x \right) dx}_A \\ & - \underbrace{\Delta t \int_{I_j} \left(\psi_h^{*n+\frac{1}{2}} (u_h^{n+\frac{1}{2}})_x + u_h^{n+\frac{1}{2}} (\psi_h^{*n+\frac{1}{2}})_x \right) dx}_B - \underbrace{\Delta t \left[(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}} - (\psi_h^{*n+\frac{1}{2}+} u_h^{n+\frac{1}{2}-})_{j+\frac{1}{2}} \right]}_C \\ & + \underbrace{\Delta t \left[(u_h^{n+\frac{1}{2}-} \psi_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}} - (u_h^{*n+\frac{1}{2}-} \psi_h^{n+\frac{1}{2}-})_{j+\frac{1}{2}} \right]}_D + \underbrace{\Delta t \left[(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}} - (\psi_h^{*n+\frac{1}{2}+} u_h^{n+\frac{1}{2}+})_{j-\frac{1}{2}} \right]}_G \\ & - \underbrace{\Delta t \left[(u_h^{n+\frac{1}{2}-} \psi_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}} - (u_h^{*n+\frac{1}{2}-} \psi_h^{n+\frac{1}{2}+})_{j-\frac{1}{2}} \right]}_H = 0. \end{aligned} \quad (2.13)$$

By the chain of rule, we can derive

$$\begin{aligned} A &= \Delta t \int_{I_j} (\psi_h^{n+\frac{1}{2}} u_h^{*n+\frac{1}{2}})_x dx = \Delta t \left[(\psi_h^{n+\frac{1}{2}-} u_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}} - (\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}} \right], \\ B &= \Delta t \int_{I_j} (\psi_h^{*n+\frac{1}{2}} u_h^{n+\frac{1}{2}})_x dx = \Delta t \left[(u_h^{n+\frac{1}{2}-} \psi_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}} - (u_h^{n+\frac{1}{2}+} \psi_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}} \right], \end{aligned}$$

then

$$A - B = 2i\Delta t \left[\text{Im}(\psi_h^{n+\frac{1}{2}-} u_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}} - \text{Im}(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}} \right]. \quad (2.14)$$

After some simple algebraic manipulation $a - a^* = 2i\text{Im}(a)$, $a \in \mathcal{C}$, we have

$$\begin{aligned} C &= 2i\Delta t \text{Im}(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}}, \quad D = -2i\Delta t \text{Im}(\psi_h^{n+\frac{1}{2}-} u_h^{*n+\frac{1}{2}-})_{j+\frac{1}{2}}, \\ G &= 2i\Delta t \text{Im}(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}+})_{j-\frac{1}{2}}, \quad H = -2i\Delta t \text{Im}(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}-})_{j-\frac{1}{2}}. \end{aligned} \quad (2.15)$$

We combine all these equalities (2.13), (2.14) and (2.15) to obtain

$$\int_{I_j} (|u_h^{n+1}|^2 - |u_h|^n) dx + \hat{\Phi}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{\Phi}_{j-\frac{1}{2}}^{n+\frac{1}{2}} = 0,$$

where the numerical entropy flux is given by

$$\hat{\Phi}^{n+\frac{1}{2}} = -2\Delta t \text{Im}(\psi_h^{n+\frac{1}{2}+} u_h^{*n+\frac{1}{2}-}).$$

Summing up over j , the flux terms vanish because of the periodic boundary conditions. Thus we finish the proof. \square

COROLLARY 2.1 The discrete charge conservation law trivially implies an L^2 -stability of the numerical solution.

3. Error estimates for the full-discrete method

In this section, we will state the error estimate of the symplectic local discontinuous Galerkin method for the problem (1.1) with $d = 1$. In the sequel, \mathbb{E} denotes an expectation operator of a random variable, and K, C are constants depending on $\|Q\|_{\mathbb{H}^3}$, the final time T and the norm of u_0 , but independent of h and n . They may change from line to line.

In order to obtain the error estimate to the symplectic local discontinuous Galerkin method (2.5) with the fluxes (2.6), we divide the error into two parts:

$$\|u(t_n) - u_h^n\|^2 \leq \underbrace{\|u(\cdot, t_n) - u^n\|^2}_{\text{Temporal error}} + \underbrace{\|u^n - u_h^n\|^2}_{\text{Spatial error}}. \quad (3.1)$$

3.1 Temporal error

To obtain the temporal error estimate, we need some regularity results of the numerical solution $u^n(x)$ for (2.4). We state it in the following two lemmas. The proof of these lemmas will be given in Appendix A and Appendix B, respectively.

LEMMA 3.1 Assume that $Q \in \mathbb{H}^\gamma$ and $\mathbb{E}\|u^0\|_{\mathbb{H}^\gamma}^{2p} < \infty$, $\gamma = 0, 1, \dots$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^\gamma)$. We have the following regularity of temporal semi-discretization, i.e., for $p \geq 1$,

$$\mathbb{E}\|u^n\|_{\mathbb{H}^\gamma}^{2p} \leq K, \quad n = 1, 2, \dots, N. \quad (3.2)$$

LEMMA 3.2 Given $\gamma = 1, 2, \dots$, and assume $Q \in \mathbb{H}^\gamma$, $u^0 \in L^{2p}(\Omega; \mathbb{H}^\gamma)$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^\gamma)$, then we have holder continuity in temporal direction, i.e., for $p \geq 1$,

$$\mathbb{E}\|u^{n+1} - u^n\|_{\mathbb{H}^{\gamma-1}}^{2p} \leq K\Delta t^p, \quad n = 1, 2, \dots, N.$$

Now we are in a position to establish an error estimate of the semi-discrete method (2.4) by virtue of these two lemmas.

THEOREM 3.1 Assume that $u_0 \in L^2(\Omega; \mathbb{H}^3)$, $Q \in \mathbb{H}^3$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^3)$ then it is of the mean-square order 1, i.e.,

$$\left(\mathbb{E}\|u(t_n) - u^n\|_{\mathbb{L}^2}^2 \right)^{1/2} \leq K\Delta t.$$

Proof. From (A.2) (see Appendix A) and (1.1), it follows

$$u^{n+1} = \hat{S}_{\Delta t}^{n+1} u^0 - i\Delta t \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q u^{\ell-\frac{1}{2}} - i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1}, \quad (3.3)$$

and

$$\begin{aligned} u(t_{n+1}) &= S(t_{n+1}) u^0 - i \int_0^{t_{n+1}} S(t_{n+1} - \tau) Q u(\tau) d\tau - i \int_0^{t_{n+1}} S(t_{n+1} - \tau) u(\tau) \circ dW(\tau) \\ &= S(t_{n+1}) u^0 - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - \tau) Q u(\tau) d\tau - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - \tau) u(\tau) \circ dW(\tau). \end{aligned} \quad (3.4)$$

Subtract (3.3) from (3.4) leads to

$$\begin{aligned} u(t_{n+1}) - u^{n+1} &= \underbrace{\left(S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1} \right) u^0}_{\mathcal{A}} - i \sum_{\ell=1}^{n+1} \left(\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - \tau) Q u(\tau) d\tau - \Delta t \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q u^{\ell-\frac{1}{2}} \right) \\ &\quad - i \sum_{\ell=1}^{n+1} \left(\int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - \tau) u(\tau) \circ dW(\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1} \right) \\ &=: \mathcal{A} + \mathcal{B} + \mathcal{C}. \end{aligned}$$

We will estimate them separately.

• **The first term \mathcal{A} .**

From (Bouard & Debussche, 2006), we know that $\|S(t_{n+1}) - \hat{S}_{\Delta t}^{n+1}\|_{\mathcal{L}(\mathbb{H}^3, \mathbb{L}^2)} \leq K\Delta t$. Thus,

$$\mathbb{E}\|\mathcal{A}\|_{\mathbb{L}^2}^2 \leq K\mathbb{E}\|u^0\|_{\mathbb{H}^3}^2 \Delta t^2 \leq K\Delta t^2.$$

• **The second term \mathcal{B} .**

To estimate \mathcal{B} , we insert one term

$$\pm i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - r) Q u_{t_{\ell-1}, u^{\ell-1}}(\tau) d\tau$$

into the expression of \mathcal{B} and we have

$$\begin{aligned}\mathcal{B} &= -i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} S(t_{n+1} - \tau) Q(u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau)) d\tau \\ &\quad - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left(S(t_{n+1} - r) Q u_{t_{\ell-1}, u^{\ell-1}}(\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q u^{\ell-\frac{1}{2}} \right) d\tau \\ &=: \mathcal{B}^1 + \mathcal{B}^2.\end{aligned}$$

To estimate term \mathcal{B}^1 , we present the estimate of $u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau)$ by their expression,

$$\begin{aligned}u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau) &= S(\tau - t_{\ell-1})(u(t_{\ell-1}) - u^{\ell-1}) \\ &\quad - i \int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1} - \rho) Q(u(\rho) - u_{t_{\ell-1}, u^{\ell-1}}(\rho)) d\rho \\ &\quad - i \int_{t_{\ell-1}}^{\tau} S(\tau - t_{\ell-1} - \rho) (u(\rho) - u_{t_{\ell-1}, u^{\ell-1}}(\rho)) \circ dW(\rho).\end{aligned}$$

Therefore, from Gronwall's inequality, we know $\mathbb{E}\|u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau)\|_{\mathbb{L}^2}^2 \leq K\mathbb{E}\|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2$, and for term \mathcal{B}^1

$$\mathbb{E}\|\mathcal{B}^1\|_{\mathbb{L}^2}^2 \leq K\Delta t \sum_{\ell=1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2.$$

We split term \mathcal{B}^2 further as follows

$$\begin{aligned}\mathcal{B}^2 &= -i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \left(S(t_{n+1} - r) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \right) Q u_{t_{\ell-1}, u^{\ell-1}}(\tau) d\tau \\ &\quad - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \left(u_{t_{\ell-1}, u^{\ell-1}}(\tau) - u^{\ell-1} \right) d\tau \\ &\quad - i\Delta t \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \left(u^{\ell} - u^{\ell-1} \right) \\ &=: \mathcal{B}_a^2 + \mathcal{B}_b^2 + \mathcal{B}_c^2.\end{aligned}$$

For term \mathcal{B}_a^2 , based on $\|S(t_n) - \hat{S}_{\Delta t}^n\|_{\mathcal{L}(\mathbb{H}^3; \mathbb{L}^2)} \leq K\Delta t$, $\|I - T_{\Delta t}\|_{\mathcal{L}(\mathbb{H}^3; \mathbb{L}^2)} \leq K\Delta t$ and Lemma 3.1, we have

$$\mathbb{E}\|\mathcal{B}_a^2\|_{\mathbb{L}^2}^2 \leq K\Delta t^2.$$

To estimate term \mathcal{B}_b^2 , we insert the expression of $u_{t_{\ell-1}, u^{\ell-1}}(\tau) - u^{\ell-1}$ into it and we have

$$\begin{aligned}\mathcal{B}_b^2 &= -i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \left[(S(\tau - t_{\ell-1}) - I) u^{\ell-1} d\tau \right. \\ &\quad \left. - i \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) \left(Q - \frac{i}{2} \right) u_{t_{\ell-1}, u^{\ell-1}}(\rho) d\rho \right] d\tau \\ &\quad - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) d\tau.\end{aligned}$$

The estimate of the first term is similar to before and is bounded by $K\Delta t^2$.

Concerning the second term, we employ Fubini's theorem and Itô isometry and Lemma 3.1,

$$\begin{aligned} & \mathbb{E} \left\| - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \int_{t_{\ell-1}}^{\tau} S(\tau-\rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) d\tau \right\|_{\mathbb{L}^2}^2 \\ &= \mathbb{E} \left\| - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} Q \int_{\rho}^{t_\ell} S(\tau-\rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) d\tau dW(\rho) \right\|_{\mathbb{L}^2}^2 \\ &\leq K\Delta t^2. \end{aligned}$$

The estimate of term \mathcal{B}_c^2 is similar to that of term \mathcal{B}_b^2 , by replacing the expression of $u^\ell - u^{\ell-1}$. Combining all the above inequalities, we obtain the desired estimate of \mathcal{B}

$$\mathbb{E} \|\mathcal{B}\|_{\mathbb{L}^2}^2 \leq K\Delta t^2 + K\Delta t \sum_{\ell=1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2.$$

• **The third term \mathcal{C} .**

To estimate \mathcal{C} , we change Stratonovich integral into Itô one with noting that $F_\phi = \sum_{\ell \in \mathbb{N}^d} (\phi e_\ell(x))^2$,

$$\begin{aligned} \mathcal{C} &= -\frac{1}{2} \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) u(\tau) F_\phi d\tau \\ &\quad - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) u(\tau) dW(\tau) + i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1}. \end{aligned}$$

We split it further

$$\begin{aligned} \mathcal{C} &= -i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) \left(u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau) \right) dW(\tau) - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \left(S(t_{n+1}-\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \right) u_{t_{\ell-1}, u^{\ell-1}}(\tau) dW(\tau) \\ &\quad - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left(u_{t_{\ell-1}, u^{\ell-1}}(\tau) - u^{\ell-1} \right) dW(\tau) + \frac{i}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left(u^\ell - u^{\ell-1} \right) \Delta \tilde{W}_{\ell-1} \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) u(\tau) F_\phi d\tau + i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \left(\Delta \tilde{W}_{\ell-1} - \Delta W_{\ell-1} \right). \end{aligned}$$

By replacing the expressions of $u_{t_{\ell-1}, u^{\ell-1}}(\tau) - u^{\ell-1}$ and $u^\ell - u^{\ell-1}$ into the above equation, we have

$$\begin{aligned} \mathcal{C} &= -i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) \left(u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\tau) \right) dW(\tau) - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \left(S(t_{n+1}-\tau) - \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \right) u_{t_{\ell-1}, u^{\ell-1}}(\tau) dW(\tau) \\ &\quad - i \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left((S(\tau-t_{\ell-1}) - I) u^{\ell-1} - i \int_{t_{\ell-1}}^{\tau} S(\tau-\rho) (Q - \frac{i}{2}) u_{t_{\ell-1}, u^{\ell-1}}(\rho) d\rho \right) dW(\tau) \\ &\quad + \frac{i}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \left((\hat{S}_{\Delta t} - I) u^{\ell-1} - i \Delta t T_{\Delta t} Q u^{\ell-\frac{1}{2}} \right) \Delta \tilde{W}_{\ell-1} - \frac{1}{2} \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) \left(u(\tau) - u_{t_{\ell-1}, u^{\ell-1}}(\rho) \right) F_\phi d\tau \\ &\quad - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} S(\tau-\rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) + \frac{1}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-\frac{1}{2}} (\Delta \tilde{W}_{\ell-1})^2 \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1}-\tau) u_{t_{\ell-1}, u^{\ell-1}}(\rho) F_\phi d\tau + i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \left(\Delta \tilde{W}_{\ell-1} - \Delta W_{\ell-1} \right). \end{aligned} \tag{3.5}$$

We pay more attention to the last three lines, denoted by \mathcal{D} , because other terms can be estimated as before, and are bounded by $K\Delta t^2 + K\Delta t \sum_{\ell=1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2$.

We have

$$\begin{aligned}
& - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} S(\tau - \rho) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\
& = - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} \left(S(\tau - \rho) - T_{\Delta t} \right) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\
& - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_{\ell}} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} T_{\Delta t} \left(u_{t_{\ell-1}, u^{\ell-1}}(\rho) - u^{\ell-1} \right) dW(\rho) dW(\tau) \\
& - \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} dW(\rho) dW(\tau).
\end{aligned}$$

We claim that the last term in the above equality has the form

$$\sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} dW(\rho) dW(\tau) = \frac{1}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \left((\Delta W_{\ell-1})^2 - F_{\phi} \Delta t \right).$$

In fact,

$$\begin{aligned}
& \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} dW(\rho) dW(\tau) \\
& = \sum_{k_1, k_2 \in \mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_2}(\tau) \\
& = \sum_{k_1=k_2 \in \mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_1}(\tau) \\
& + \sum_{k_1 < k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_2}(\tau) \\
& + \sum_{k_1 > k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_2}(\tau) \\
& = I + II + III.
\end{aligned} \tag{3.6}$$

Due to $\int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_1}(\tau) = \frac{1}{2} \left((\Delta \beta_{k_1})^2 - \Delta t F_{\phi} \right)$, we have

$$I = \frac{1}{2} \sum_{k_1=k_2 \in \mathbb{N}} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \left((\Delta \beta_{k_1})^2 - \Delta t F_{\phi} \right).$$

We change the index of k_1 and k_2 in the last term of (3.6) to obtain

$$III = \sum_{k_2 > k_1} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_2} \phi e_{k_1} \int_{t_{\ell-1}}^{t_{\ell}} \int_{t_{\ell-1}}^{\tau} d\beta_{k_2}(\rho) d\beta_{k_1}(\tau),$$

and

$$\begin{aligned} II + III &= \sum_{k_1 < k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \left[\int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} d\beta_{k_1}(\rho) d\beta_{k_2}(\tau) + \int_{t_{\ell-1}}^{t_\ell} \int_{t_{\ell-1}}^{\tau} d\beta_{k_2}(\rho) d\beta_{k_1}(\tau) \right] \\ &= \sum_{k_1 < k_2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} \phi e_{k_1} \phi e_{k_2} \Delta \beta_{k_1} \Delta \beta_{k_2}. \end{aligned}$$

Combining them together we may prove the claim.

It follows from the rearrangement of the last three lines of (3.5) that,

$$\begin{aligned} \mathcal{D} &= - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} (S(\tau - \rho) - T_{\Delta t}) u_{t_{\ell-1}, u^{\ell-1}}(\rho) dW(\rho) dW(\tau) \\ &\quad - \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} \int_{t_{\ell-1}}^{\tau} T_{\Delta t} (u_{t_{\ell-1}, u^{\ell-1}}(\rho) - u^{\ell-1}) dW(\rho) dW(\tau) \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} ((\Delta W_{\ell-1})^2 - (\Delta \hat{W}_{\ell-1})^2) + \frac{1}{4} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 (u^\ell - u^{\ell-1}) (\Delta \hat{W}_{\ell-1})^2 \\ &\quad - \frac{1}{2} \sum_{\ell=1}^{n+1} \int_{t_{\ell-1}}^{t_\ell} S(t_{n+1} - \tau) u_{t_{\ell-1}, u^{\ell-1}}(\rho) F_\phi d\tau + \frac{1}{2} \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t}^2 u^{\ell-1} F_\phi \Delta t \\ &\quad + i \sum_{\ell=1}^{n+1} \hat{S}_{\Delta t}^{n+1-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} (\Delta \tilde{W}_{\ell-1} - \Delta W_{\ell-1}). \end{aligned}$$

The estimates of the first two lines come from Itô isometry, and are bounded by $K\Delta t^2$. By the properties of the truncated Wiener process, the estimate of the last line is similar to that of \mathcal{B}^2 , and is bounded also by $K\Delta t^2$.

Combing all these analysis above, we obtain

$$\mathbb{E} \|u(t_{n+1}) - u^{n+1}\|_{\mathbb{L}^2}^2 \leq K\Delta t^2 + K\Delta t \sum_{\ell=1}^{n+1} \|u(t_{\ell-1}) - u^{\ell-1}\|_{\mathbb{L}^2}^2.$$

Therefore, Gronwall's lemma leads to the assertion. \square

3.2 Spatial error

We state the spatial error estimate of the symplectic local discontinuous Galerkin method (2.5) for the stochastic linear Schrödinger equation (1.1).

THEOREM 3.2 Assume $u_0 \in L^2(\Omega; \mathbb{H}^{k+2})$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^{k+2})$. Let u_h^n be the numerical solution of the symplectic local discontinuous Galerkin method (2.5). Then there exists a constant $h_0 > 0$ such that for $h \leq h_0$,

$$\mathbb{E} \|u^n - u_h^n\|_{\mathbb{L}^2}^2 \leq Ch^{2k+2} + C\Delta t^{-1} h^{2k+2}. \quad (3.7)$$

Proof. We split the proof into two steps:

Step 1: The error equation.

Notice that the method (2.5) is also satisfied when the numerical solutions r_h, p_h, s_h, q_h are replaced by the exact solutions $r, p = s_x, s, q = s_x$. For each fixed t_n , we can obtain the cell error equation

$$\begin{aligned}
& \mathfrak{B}_j(r^n - r_h^n, p^n - p_h^n, s^n - s_h^n, q^n - q_h^n; v_h, \omega_h, \alpha_h, \beta_h) \\
&= \int_{I_j} [r^{n+1} - r_h^{n+1}] v_h dx - \int_{I_j} [r^n - r_h^n] v_h dx + \Delta t \int_{I_j} (p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}) (v_h)_x dx \\
&- \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}}) v_h \Delta \tilde{W}_n dx - \Delta t \int_{I_j} (p^{n+\frac{1}{2}} - p_h^{n+\frac{1}{2}}) \omega_h dx - \Delta t \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}}) (\omega_h)_x dx \\
&- \Delta t \int_{I_j} (s^{n+\frac{1}{2}} - s_h^{n+\frac{1}{2}}) Q_h v_h dx + \Delta t \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}}) Q_h \alpha_h dx \\
&- \Delta t \int_{I_j} (q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}}) (\alpha_h)_x dx + \int_{I_j} [s^{n+1} - s_h^{n+1}] \alpha_h dx - \int_{I_j} [s^n - s_h^n] \alpha_h dx \\
&+ \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}}) \alpha_h \Delta \tilde{W}_n dx - \Delta t \int_{I_j} (q^{n+\frac{1}{2}} - q_h^{n+\frac{1}{2}}) \beta_h dx - \Delta t \int_{I_j} (r^{n+\frac{1}{2}} - r_h^{n+\frac{1}{2}}) (\beta_h)_x dx \\
&- \Delta t [(p^{n+\frac{1}{2}} - \hat{p}^{n+\frac{1}{2}}) v_h^-]_{j+\frac{1}{2}} + \Delta t [(p^{n+\frac{1}{2}} - \hat{p}^{n+\frac{1}{2}}) v_h^+]_{j-\frac{1}{2}} + \Delta t [(s^{n+\frac{1}{2}} - \hat{s}^{n+\frac{1}{2}}) \omega_h^-]_{j+\frac{1}{2}} \\
&- \Delta t [(s^{n+\frac{1}{2}} - \hat{s}^{n+\frac{1}{2}}) \omega_h^+]_{j-\frac{1}{2}} + \Delta t [(q^{n+\frac{1}{2}} - \hat{q}^{n+\frac{1}{2}}) \alpha_h^-]_{j+\frac{1}{2}} - \Delta t [(q^{n+\frac{1}{2}} - \hat{q}^{n+\frac{1}{2}}) \alpha_h^+]_{j-\frac{1}{2}} \\
&+ \Delta t [(r^{n+\frac{1}{2}} - \hat{r}^{n+\frac{1}{2}}) \beta_h^-]_{j+\frac{1}{2}} - \Delta t [(r^{n+\frac{1}{2}} - \hat{r}^{n+\frac{1}{2}}) \beta_h^+]_{j-\frac{1}{2}} = 0
\end{aligned} \tag{3.8}$$

for all $v_h, \omega_h, \alpha_h, \beta_h \in V_h^k$.

Summing over j , the error equation becomes

$$\sum_{j=1}^J \mathfrak{B}_j(r^n - r_h^n, p^n - p_h^n, s^n - s_h^n, q^n - q_h^n; v_h, \omega_h, \alpha_h, \beta_h) = 0 \tag{3.9}$$

for all $v_h, \omega_h, \alpha_h, \beta_h \in V_h^k$.

Denoting

$$\begin{aligned}
& \varepsilon^n = \mathcal{P}^- r^n - r_h^n, \quad \xi^n = \mathcal{P} q^n - q_h^n, \quad \eta^n = \mathcal{P}^- s^n - s_h^n, \quad \zeta^n = p_h^n - \mathcal{P} p^n, \\
& \varepsilon_e^n = \mathcal{P}^- r^n - r^n, \quad \xi_e^n = \mathcal{P} q^n - q^n, \quad \eta_e^n = \mathcal{P}^- s^n - s^n, \quad \zeta_e^n = p^n - \mathcal{P} p^n,
\end{aligned} \tag{3.10}$$

where \mathcal{P} is the standard \mathbb{L}^2 -projection of a function ω with $k+1$ continuous derivatives into space V_h^k , \mathcal{P}^- is a special projector into V_h^k , which satisfies, for each j ,

$$\int_{I_j} (\mathcal{P}^- \omega(x) - \omega(x)) v(x) dx = 0, \quad \forall v \in P^{k-1}(I_j),$$

and $\mathcal{P}^-(\omega(x_{j+\frac{1}{2}}^-)) = \omega(x_{j+\frac{1}{2}})$. and taking the test functions

$$v_h = \varepsilon^{n+\frac{1}{2}}, \quad \omega_h = \xi^{n+\frac{1}{2}}, \quad \alpha_h = \eta^{n+\frac{1}{2}}, \quad \beta_h = \zeta^{n+\frac{1}{2}},$$

we obtain the important energy equality

$$\sum_{j=1}^J \mathfrak{B}_j(\varepsilon^n - \varepsilon_e^n, \zeta_e^n - \zeta^n, \eta^n - \eta_e^n, \xi^n - \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) = 0. \tag{3.11}$$

Now, we shall prove the theorem by analyzing each terms of (3.11).

Step 2: Proof of the main result.

We consider the left-hand side of the energy equation (3.11). Using the linearity of \mathfrak{B}_j with respect to its first group of arguments, we get

$$\begin{aligned} & \mathfrak{B}_j(\varepsilon^n - \varepsilon_e^n, \zeta_e^n - \zeta^n, \eta^n - \eta_e^n, \xi^n - \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &= \mathfrak{B}_j(\varepsilon^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) - \mathfrak{B}_j(\varepsilon_e^n, -\zeta_e^n, \eta_e^n, \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}). \end{aligned} \quad (3.12)$$

First, we consider the first term of the right-hand side in (3.12), which yields

$$\begin{aligned} & \mathfrak{B}_j(\varepsilon^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &= \frac{1}{2} \int_{I_j} ((\varepsilon^{n+1})^2 - (\varepsilon^n)^2) dx + \frac{1}{2} \int_{I_j} ((\eta^{n+1})^2 - (\eta^n)^2) dx \\ &+ \Delta t [(\zeta^+ \varepsilon^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\zeta^+ \varepsilon^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}] + \Delta t [(\eta^- \xi^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\eta^- \xi^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}] \\ &+ \Delta t [(\xi^+ \eta^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\xi^+ \eta^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}] + \Delta t [(\varepsilon^- \zeta^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\varepsilon^- \zeta^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}}] \\ &- \Delta t \underbrace{\int_{I_j} [(\eta \xi)_x^{n+\frac{1}{2}} + (\varepsilon \zeta)_x^{n+\frac{1}{2}}] dx}_R. \end{aligned} \quad (3.13)$$

From the integration by parts, we arrive at

$$R = \left[(\eta^- \xi^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\eta^+ \xi^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right] + \left[(\varepsilon^- \zeta^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\varepsilon^+ \zeta^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right]. \quad (3.14)$$

Substituting (3.14) into (3.13), we have

$$\begin{aligned} & \mathfrak{B}_j(\varepsilon^n, -\zeta^n, \eta^n, \xi^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) \\ &= \frac{1}{2} \int_{I_j} ((\varepsilon^{n+1})^2 - (\varepsilon^n)^2) dx + \frac{1}{2} \int_{I_j} ((\eta^{n+1})^2 - (\eta^n)^2) dx + \Delta t [\hat{\Phi}_{j+\frac{1}{2}}^{n+\frac{1}{2}} - \hat{\Phi}_{j-\frac{1}{2}}^{n+\frac{1}{2}}], \end{aligned} \quad (3.15)$$

where $\hat{\Phi} = \xi^+ \eta^- + \zeta^+ \varepsilon^-$.

As for the second term of the right-hand side in (3.12), we have

$$\mathfrak{B}_j(\varepsilon_e^n, -\zeta_e^n, \eta_e^n, \xi_e^n; \varepsilon^{n+\frac{1}{2}}, \xi^{n+\frac{1}{2}}, \eta^{n+\frac{1}{2}}, \zeta^{n+\frac{1}{2}}) = I + II + III + IV + V, \quad (3.16)$$

where

$$\begin{aligned} I &= \int_{I_j} (\varepsilon_e^{n+1} - \varepsilon_e^n) \varepsilon^{n+\frac{1}{2}} dx + \int_{I_j} (\eta_e^{n+1} - \eta_e^n) \eta^{n+\frac{1}{2}} dx, \\ II &= \Delta t \int_{I_j} \left((\zeta_e \xi)^{n+\frac{1}{2}} - (\zeta_e \varepsilon_x)^{n+\frac{1}{2}} - (\eta_e \xi_x)^{n+\frac{1}{2}} - (\xi_e \eta_x)^{n+\frac{1}{2}} \right. \\ &\quad \left. - (\xi_e \zeta)^{n+\frac{1}{2}} - (\varepsilon_e \zeta_x)^{n+\frac{1}{2}} - Q_h(\eta_e \varepsilon)^{n+\frac{1}{2}} + Q_h(\varepsilon_e \eta)^{n+\frac{1}{2}} \right) dx, \end{aligned}$$

$$\begin{aligned}
III &= \int_{I_j} \varepsilon_e^{n+\frac{1}{2}} \eta^{n+\frac{1}{2}} \Delta \tilde{W}_n dx, \quad IV = - \int_{I_j} \eta_e^{n+\frac{1}{2}} \varepsilon^{n+\frac{1}{2}} \Delta \tilde{W}_n dx, \\
V &= \Delta t \left[(\zeta_e^+ \varepsilon^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\zeta_e^+ \varepsilon^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} - (\eta_e^- \xi^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} + (\eta_e^- \xi^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right. \\
&\quad \left. - (\xi_e^e \eta^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} - (\xi_e^+ \eta^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} - (\varepsilon_e^- \zeta^-)_{j+\frac{1}{2}}^{n+\frac{1}{2}} + (\varepsilon_e^- \zeta^+)_{j-\frac{1}{2}}^{n+\frac{1}{2}} \right].
\end{aligned}$$

By using the simple inequality $ab \leq \frac{a^2}{4} + b^2$, and the standard approximation theory (3.17) on ε^e , and η^e , we have

$$\begin{aligned}
I &\leq \|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2(I_j)} \|\varepsilon^{n+\frac{1}{2}}\|_{\mathbb{L}^2(I_j)} + \|\eta_e^{n+1} - \eta_e^n\|_{\mathbb{L}^2(I_j)} \|\eta^{n+\frac{1}{2}}\|_{\mathbb{L}^2(I_j)} \\
&\leq C\Delta t^{-1} \|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2(I_j)}^2 + C\Delta t \|\varepsilon^{n+\frac{1}{2}}\|_{\mathbb{L}^2(I_j)}^2 + C\Delta t^{-1} \|\eta_e^{n+1} - \eta_e^n\|_{\mathbb{L}^2(I_j)}^2 + C\Delta t \|\eta^{n+\frac{1}{2}}\|_{\mathbb{L}^2(I_j)}^2,
\end{aligned}$$

where $\|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2(I_j)} = \|\mathcal{P}^-(r^{n+1} - r^n) - (r^{n+1} - r^n)\|_{\mathbb{L}^2(I_j)}$ and $\|\eta_e^{n+1} - \eta_e^n\|_{\mathbb{L}^2(I_j)} = \|\mathcal{P}^-(s^{n+1} - s^n) - (s^{n+1} - s^n)\|_{\mathbb{L}^2(I_j)}$. It is well know that for any $\omega \in \mathbb{H}^{k+1}(\mathbb{R})$

$$\|\check{\omega}(x)\|_{\mathbb{L}^2} + h\|\check{\omega}(x)\|_{\mathbb{L}^\infty} + \sqrt{h}\|\check{\omega}(x)\|_{\Gamma_h} \leq C\|\omega\|_{\mathbb{H}^{k+1}} h^{k+1} \quad (3.17)$$

where $\check{\omega} = \mathcal{P}\omega - \omega$ or $\check{\omega} = \mathcal{P}^-\omega - \omega$. The positive constant C is independent of h , and Γ_h is the usual L^2 -norm on the cell interfaces of the mesh, which for this one-dimensional case is $\|v\|_{\Gamma_h}^2 = \sum_{j=1}^J \left((v_{j+\frac{1}{2}}^-)^2 + (v_{j-\frac{1}{2}}^+)^2 \right)$.

Summing over j and taking expectation, utilizing the property of projection and the estimate of $\mathbb{E}\|r^{n+1} - r^n\|_{\mathbb{H}^{k+1}}^2$ (see Lemma 3.2 with $p = 1$) and Lemma 3.1, we have

$$\mathbb{E}\left(\sum_{j=1}^J I\right) \leq C\mathbb{E}\|u_0\|_{\mathbb{H}^{k+2}}^2 h^{2k+2} + C\Delta t \mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2 + C\Delta t \mathbb{E}\|\eta^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2. \quad (3.18)$$

From the property of the projections \mathcal{P} and \mathcal{P}^- , it follows that all the terms in II except the last two terms are actually zero. We can get the estimates for II via Young's inequality and Lemma 3.1,

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^J II\right) &\leq C\mathbb{E}(\|r^n\|_{\mathbb{H}^{k+2}}^2 + \|s^n\|_{\mathbb{H}^{k+2}}^2) \Delta t h^{2k+2} + \frac{\Delta t}{4} \mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2 + \frac{\Delta t}{4} \mathbb{E}\|\eta^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2 \\
&\leq C\mathbb{E}\|u_0\|_{\mathbb{H}^{k+2}}^2 \Delta t h^{2k+2} + \frac{\Delta t}{4} \mathbb{E}\|\varepsilon^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2 + \frac{\Delta t}{4} \mathbb{E}\|\eta^{n+\frac{1}{2}}\|_{\mathbb{L}^2([L_f, L_r])}^2
\end{aligned}$$

For the third term III , we have

$$\begin{aligned}
\mathbb{E}\left(\sum_{j=1}^J III\right) &= \frac{1}{4} \mathbb{E} \int_{L_f}^{L_r} (\varepsilon_e^{n+1} - \varepsilon_e^n) (\eta^{n+1} - \eta^n) \Delta \tilde{W}_n dx \\
&\quad + \frac{1}{2} \mathbb{E} \int_{L_f}^{L_r} (\varepsilon_e^{n+1} - \varepsilon_e^n) \eta^n \Delta \tilde{W}_n dx + \frac{1}{2} \mathbb{E} \int_{L_f}^{L_r} \varepsilon_e^n (\eta^{n+1} - \eta^n) \Delta \tilde{W}_n dx \\
&=: III^a + III^b + III^c.
\end{aligned}$$

For term III^a , using Young's inequality, Lemma 3.1 and Lemma 3.2 with $p = 2$, we have

$$\begin{aligned}
 III^a &\leq \frac{1}{4} \mathbb{E} \left(\|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])} \|\eta^{n+1} - \eta^n\|_{\mathbb{L}^2([L_f, L_r])} \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])} \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^{n+1} - \eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t^{-1} \mathbb{E} \left(\|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])}^2 \|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^{n+1} - \eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t^{-1} \left(h^{2k+2} \mathbb{E} \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])}^4 + h^{-(2k+2)} \mathbb{E} \|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^4 \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t \mathbb{E} \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t h^{2k+2}.
 \end{aligned}$$

Similarly, for term III^b ,

$$\begin{aligned}
 III^b &\leq \frac{1}{2} \mathbb{E} \left(\|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])} \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])} \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])} \right) \\
 &\leq C \mathbb{E} \|\varepsilon_e^{n+1} - \varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \mathbb{E} \left(\|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])}^2 \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t h^{2k+2},
 \end{aligned}$$

and for term III^c ,

$$\begin{aligned}
 III^c &\leq \frac{1}{2} \mathbb{E} \left(\|\varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])} \|\eta^{n+1} - \eta^n\|_{\mathbb{L}^2([L_f, L_r])} \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])} \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^{n+1} - \eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t^{-1} \mathbb{E} \left(\|\varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \|\Delta \tilde{W}_n\|_{\mathbb{L}^\infty([L_f, L_r])}^2 \right) \\
 &\leq C \Delta t \mathbb{E} \|\eta^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t \mathbb{E} \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C h^{2k+2},
 \end{aligned}$$

where in the last inequalities for the estimate of III^b and III^c , we use the independent property of Wiener process. The estimate of term IV is similar as that of term III , so we omit the process here.

Finally, V only contains flux difference terms which all vanish upon a summation in j . Combining these together, we know that

$$\begin{aligned}
 &\frac{1}{2} \mathbb{E} \left(\|\varepsilon_e^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 + \|\eta^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 \right) - \frac{1}{2} \mathbb{E} \left(\|\varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \right) \\
 &\leq C \Delta t \mathbb{E} \|\varepsilon_e^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t \mathbb{E} \|\varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \\
 &\quad + C \Delta t \mathbb{E} \|\eta^{n+1}\|_{\mathbb{L}^2([L_f, L_r])}^2 + C \Delta t \mathbb{E} \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + C h^{2k+2} + C \Delta t h^{2k+2}.
 \end{aligned}$$

By Gronwall's inequality, there exists a constant $h_0 > 0$, for $h \leq h_0$, we obtain

$$\mathbb{E} \left(\|\varepsilon_e^n\|_{\mathbb{L}^2([L_f, L_r])}^2 + \|\eta^n\|_{\mathbb{L}^2([L_f, L_r])}^2 \right) \leq C h^{2k+2} + C \Delta t^{-1} h^{2k+2}, \quad \forall n.$$

I.e.,

$$\mathbb{E} \|u^n - u_h^n\|_{\mathbb{L}^2}^2 \leq C h^{2k+2} + C \Delta t^{-1} h^{2k+2}. \quad (3.19)$$

The proof is finished. \square

3.3 Main result

Combining Theorem 3.1 and Theorem 3.2, we obtain the error estimate of (2.5).

THEOREM 3.3 Let $u(x, t)$ be the exact solution of the problem (1.1), and assume the initial value $u_0(x) \in L^2(\Omega; \mathbb{H}^{k+2})$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^{k+2})$ ($k \geq 1$). Let u_h^n be the numerical solution of the symplectic local discontinuous Galerkin method (2.5). Then there exists a constant $h_0 > 0$ such that for $h \leq h_0$ holds

$$\mathbb{E}\|u(t_n) - u_h^n\|_{\mathbb{L}^2}^2 \leq C\Delta t^2 + Ch^{2k+2} + C\Delta t^{-1}h^{2k+2}. \quad (3.20)$$

The overall convergence rate is usually expressed in terms of the computational cost of the scheme (Jentzen & Kloeden, 2011). Here the computational cost of method (2.5) is denoted by $M = N \cdot J$ with N and J being the total grid number in temporal and spacial directions, respectively. In view of the above error bound, it is optimal to choose $N = M^{\frac{2k+2}{2k+5}}$ and $J = M^{\frac{3}{2k+5}}$, i.e., $\Delta t = O(\frac{1}{N}) = O\left(\left(\frac{1}{M}\right)^{\frac{2k+2}{2k+5}}\right)$ and $h = O(\frac{1}{J}) = O\left(\left(\frac{1}{M}\right)^{\frac{3}{2k+5}}\right)$, and we have the optimal error bound

$$\left(\mathbb{E}\|u(t_n) - u_h^n\|_{\mathbb{L}^2}^2\right)^{\frac{1}{2}} \leq C\left(\frac{1}{M}\right)^{\frac{2k+2}{2k+5}}.$$

REMARK 3.1 If $k = 1$, i.e., the initial data $u_0 \in L^2(\Omega; \mathbb{H}^3)$ and $\phi \in \mathcal{L}_2(\mathbb{L}^2; \mathbb{H}^3)$, then the mean-square convergence rate of the method (2.5) with respect to the computational cost is $\frac{4}{7}$.

REMARK 3.2 In the section 3, the mean-square convergence was derived for the symplectic local discontinuous Galerkin method (2.5) discretized equation (1.1). Note that (1.1) is the linear Schrödinger equation. As for nonlinear equation, truncation strategy may be needed to deal with the nonlinear term, as in (Bouard & Debussche, 2004, 2006; Liu, 2013). However, things are a bit technical for the error estimation of the symplectic local discontinuous Galerkin method, since if we employ truncated strategy, then it has to start by taking \mathbb{H}^γ -norm ($\gamma > \frac{d}{2}$) on the error equation; see Remark 3.2 in (Liu, 2013). It looks like other technical strategy is needed to derive the mean-square convergence for symplectic local discontinuous Galerkin method applied to nonlinear case, and it will be our future work.

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Appendix A. Proof of lemma 3.1

Proof. We present the proof for $p = 1$ in the following, and the general case follows similarly. First of all, we rewrite temporal semi-discretization system (2.4) into the function of u^n :

$$u^{n+1} = \hat{S}_{\Delta t} u^n - i\Delta t T_{\Delta t} Q u^{n+\frac{1}{2}} - iT_{\Delta t} u^{n+\frac{1}{2}} \Delta \tilde{W}_n, \quad (\text{A.1})$$

where u^n denotes the complex function $r^n + is^n$, operators are defined by $\hat{S}_{\Delta t} = (I + i\frac{\Delta t}{2} \partial_{xx})^{-1} (I - i\frac{\Delta t}{2} \partial_{xx})$ and $T_{\Delta t} = (I + i\frac{\Delta t}{2} \partial_{xx})^{-1}$, where I is an identity operator.

It is easy to check that the operator $\hat{S}_{\Delta t}$ is isometry in \mathbb{L}^2 , i.e., $\|\hat{S}_{\Delta t}\|_{\mathcal{L}(\mathbb{L}^2; \mathbb{L}^2)} = 1$. Furthermore, we know that $\|T_{\Delta t}\|_{\mathcal{L}(\mathbb{L}^2; \mathbb{L}^2)} \leq 1$. See reference (Bouard & Debussche, 2006) for example.

Next, we replace the function of u^n into equation (A.1) iteratively. We obtain

$$u^n = \hat{S}_{\Delta t}^n u^0 - i\Delta t \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} Q u^{\ell-\frac{1}{2}} - i \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1}. \quad (\text{A.2})$$

In order to bound function u^n , we insert the equality $u^{\ell-\frac{1}{2}} = \frac{1}{2}(\hat{S}_{\Delta t} + I)u^{\ell-1} + \frac{1}{2}(u^\ell - \hat{S}_{\Delta t}u^{\ell-1})$ into the stochastic term and take \mathbb{H}^γ -norm to get

$$\begin{aligned} \|u^n\|_{\mathbb{H}^\gamma}^2 &\leq K \|u^0\|_{\mathbb{H}^\gamma}^2 + K\Delta t \sum_{\ell=1}^n \|u^{\ell-\frac{1}{2}}\|_{\mathbb{H}^\gamma}^2 + K \left\| \frac{i}{2} \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} (\hat{S}_{\Delta t} + I) u^{\ell-1} \Delta \tilde{W}_{\ell-1} \right\|_{\mathbb{H}^\gamma}^2 \\ &\quad + Kn \sum_{\ell=1}^n \|(u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1}\|_{\mathbb{H}^\gamma}^2. \end{aligned} \quad (\text{A.3})$$

For the third term on the right-hand side of (A.3), using the fact that $u^{\ell-1}$ is independent of increment $\Delta \tilde{W}_{\ell-1}$, we have

$$\begin{aligned} \mathbb{E} \left\| \frac{i}{2} \sum_{\ell=1}^n \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} (\hat{S}_{\Delta t} + I) u^{\ell-1} \Delta \tilde{W}_{\ell-1} \right\|_{\mathbb{H}^\gamma}^2 &= \sum_{\ell=1}^n \mathbb{E} \left\| \hat{S}_{\Delta t}^{n-\ell} T_{\Delta t} (\hat{S}_{\Delta t} + I) u^{\ell-1} \Delta \tilde{W}_{\ell-1} \right\|_{\mathbb{H}^\gamma}^2 \\ &\leq K\Delta t \sum_{\ell=1}^n \mathbb{E} \|u^{\ell-1}\|_{\mathbb{H}^\gamma}^2. \end{aligned} \quad (\text{A.4})$$

To estimate the last term on the right-hand side of (A.3), we note that

$$\begin{aligned} (u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1} &= -i\Delta t T_{\Delta t} Q u^{\ell-\frac{1}{2}} \Delta \tilde{W}_{\ell-1} - \frac{i}{2} T_{\Delta t} (\hat{S}_{\Delta t} + I) u^{\ell-1} (\Delta \tilde{W}_{\ell-1})^2 \\ &\quad - \frac{i}{2} T_{\Delta t} \left((u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1} \right) \Delta \tilde{W}_{\ell-1}. \end{aligned} \quad (\text{A.5})$$

Taking $L^2(\Omega; \mathbb{H}^\gamma)$ -norm to obtain

$$\begin{aligned} \mathbb{E} \|(u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1}\|_{\mathbb{H}^\gamma}^2 &\leq K\Delta t^2 (\Delta t \kappa^2) \mathbb{E} \|u^{\ell-\frac{1}{2}}\|_{\mathbb{H}^\gamma}^2 + K\Delta t^2 \mathbb{E} \|u^{\ell-1}\|_{\mathbb{H}^\gamma}^2 \\ &\quad + K(\Delta t \kappa^2) \mathbb{E} \|(u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1}\|_{\mathbb{H}^\gamma}^2, \end{aligned} \quad (\text{A.6})$$

where we use the embedding $\mathbb{H}^1 \hookrightarrow \mathbb{L}^\infty$ for $\gamma = 0$, or use $\|fg\|_{\mathbb{H}^\gamma} \leq K\|f\|_{\mathbb{H}^\gamma}\|g\|_{\mathbb{H}^\gamma}$ for $\gamma \geq 1$. Note that there exists a constant $\Delta t^* > 0$ such that $K(\Delta t \kappa^2) \leq \frac{1}{2} < 1$ for $\Delta t \leq \Delta t^*$ (here K is the same as the last term on the right-hand side of (A.6)), which leads to

$$\frac{1}{2} \mathbb{E} \|(u^\ell - \hat{S}_{\Delta t} u^{\ell-1}) \Delta \tilde{W}_{\ell-1}\|_{\mathbb{H}^\gamma}^2 \leq K\Delta t^2 \left(\mathbb{E} \|u^\ell\|_{\mathbb{H}^\gamma}^2 + \mathbb{E} \|u^{\ell-1}\|_{\mathbb{H}^\gamma}^2 \right). \quad (\text{A.7})$$

Combining inequalities (A.3), (A.4) and (A.7) together, we have

$$\mathbb{E}\|u^n\|_{\mathbb{H}^\gamma}^2 \leq K + K\Delta t \sum_{\ell=0}^n \mathbb{E}\|u^\ell\|_{\mathbb{H}^\gamma}^2,$$

where the positive constant K depends on p , T , the L^2 -norm of operator $T_{\Delta t}$, $\|u^0\|_{H^\gamma}$, but not depends on Δt . The discrete Gronwall's lemma leads to the assertion. \square

Appendix B. Proof of lemma 3.2

Proof. We present the proof for $p = 1$. The estimation is similar as the proof of the last term on the right-hand side of (A.3); see estimations (A.5)-(A.7). Start from equation (A.1),

$$u^{n+1} - u^n = (\hat{S}_{\Delta t} - I)u^n - i\Delta t T_{\Delta t} Q u^{n+\frac{1}{2}} - iT_{\Delta t} u^{n+\frac{1}{2}} \Delta \tilde{W}_n.$$

Since $\|\hat{S}_{\Delta t} - I\|_{\mathcal{L}(\mathbb{H}^\gamma, \mathbb{H}^{\gamma-1})} \leq K\Delta t^{\frac{1}{2}}$, we take $L^2(\Omega; \mathbb{H}^{\gamma-1})$ -norm on both sides of the above equation and get

$$\begin{aligned} \mathbb{E}\|u^{n+1} - u^n\|_{\mathbb{H}^{\gamma-1}}^2 &\leq K\Delta t \mathbb{E}\|u^n\|_{\mathbb{H}^\gamma}^2 + K\Delta t^2 \mathbb{E}\left(\|u^n\|_{\mathbb{H}^{\gamma-1}}^2 + \|u^{n+1}\|_{\mathbb{H}^{\gamma-1}}^2\right) \\ &\quad + K\Delta t \mathbb{E}\|u^n\|_{\mathbb{H}^{\gamma-1}}^2 + K(\Delta t \kappa^2) \mathbb{E}\|u^{n+1} - u^n\|_{\mathbb{H}^{\gamma-1}}^2. \end{aligned} \tag{A.8}$$

there exists a constant $\Delta t^* > 0$ such that $K(\Delta t \kappa^2) \leq \frac{1}{2} < 1$ for $\Delta t \leq \Delta t^*$ (here K is the same as the last term on the right-hand side of (A.8)), which leads to

$$\frac{1}{2} \mathbb{E}\|u^{n+1} - u^n\|_{\mathbb{H}^{\gamma-1}}^2 \leq K\Delta t \mathbb{E}\|u^n\|_{\mathbb{H}^\gamma}^2 \leq K\Delta t.$$

This completes the proof. \square